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Duffin–Kemmer algebras revisited

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Abstract. Duffin–Kemmer algebras are studied from a modern perspective. Complete descriptions of these algebras and their simple modules are given in terms of tensor and exterior algebras. The approach is self-contained and no reference to general results on Jordan algebras and their representation theory is required. Absolute detail is provided for the more specific examples of the Duffin–Kemmer real algebras $D(q+2,q)$ (for $q = 0$, 1, and 2) which are relevant for applications in physics. A faithful representation of $D(q+2,q)$ is given in the space of real $2^{q+1} \times 2^{q+1}$ matrices; it is completely reducible and yields with multiplicity one all the irreducible representations of $D(q+2,q)$. The representation space has a natural orthogonal structure, $(\cdot,\cdot)_q$, of signature $(2^{2q+1}, 2^{2q+1})$, for $q > 0$, and $(4, 0)$, for $q = 0$. It corresponds to the bilinear form induced by the spin group, $Spin(q+2,q)$, on the tensor product space, $W(q+2,q) \otimes W(q+2,q)$, of two copies of the fundamental module of the Clifford algebra, $C(q+2,q)$. Explicit computations are made simple by establishing a one-to-one correspondence with the space of $2^q \times 2^q$ matrices with quaternion coefficients.

1. Introduction

Duffin–Kemmer (DK) algebras have been known since the late 1930s. In [1] Duffin gave the formal algebraic properties that define a particular algebra, $D(3,1)$, associated with the Lorentz group, $O(3,1)$; he claimed that there were three irreducible representations of it occurring in dimensions one, five and ten, respectively. Kemmer studied these representations in more detail [6], and a few years later [7], generalized the algebraic relations of Duffin, thus giving rise to the *DK algebra*, $D(n,C)$, associated with the orthogonal group, $O(n,C)$. Using some properties of the hypergeometric function, Kemmer proved that $D(n,C)$ has dimension $\binom{2n+1}{n}$. Moreover, he essentially proved that $D(n,C)$ is semisimple and gave a list of the dimensions of the simple modules. He recognized that the $D(n,C)$ irreducibles were linked to representations of the Clifford algebra, $C(n,C)$, of $O(n,C)$. Concrete examples, however, were only given for $D(3,1)$, and $D(4,C)$, with 10×10 and 5×5 matrices written down for the non-trivial simple representations. Different applications were given by Harish-Chandra [2], who used these representations to write Maxwell's equations in 'Dirac form' [3]. More than 20 years later, Jacobson [5] recognized that Clifford, as well as DK algebras, could be respectively obtained as *a unitary special universal envelope*, and *the unital universal multiplication envelope*, of the Jordan algebra associated with a vector space V on which a symmetric bilinear form is defined. In this way, Jacobson was the one who finally elucidated the nature of the relationship between Clifford and DK algebras through his general theory of representations of Jordan algebras.

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The purpose of this paper is twofold. First, to review the theory of DK algebras from a modern point of view, giving a self-contained exposition in the manner in which one usually approaches the theory of Clifford algebras, i.e. independently of its known relationship to Jordan algebras. We shall be able to give a geometrically appealing description of their simple modules. This is done in general, starting from first principles, and using only elementary properties of the tensor and exterior algebras (section 2). Second, to study in more detail some specific examples which are relevant in physics, namely the real algebras $D(q+2, q)$ (for $q = 0$, 1, and 2), and the geometric structure of their simple modules induced by the spin group, $Spin(q+2, q)$ (section 4). In doing so, the signatures of some useful symmetric bilinear forms on the space of matrices are obtained (section 5).

The results are as follows. Let V be a finite-dimensional vector space and let B be a non-degenerate symmetric bilinear form on V. The DK algebra, $D(V, B)$, of the pair (V, B) is the tensor algebra of V, modulo the ideal generated by the elements of the form, $uvu - B(u, v)u$, with u, and v in V. It is a finite-dimensional semisimple algebra. The simple modules of $D(V, B)$ may be realized inside the exterior algebra, $\wedge(V)$, of V: if $\dim V = 2m$, the subspaces $M_k = \wedge^k(V) \oplus \wedge^{k+1}(V)$, with $k = 0, 1, \ldots, m$, together with $\wedge^{2m}(V)$, give the complete list of irreducibles. If $\dim V = 2m+1$, the list is given by $\wedge^{2m+1}(V)$, the M_ks for $k = 0, 1, \ldots, m-1$, and the two eigenspaces of Hodge's star operator in M_m. Furthermore, the extension of B to $\wedge(V)$ induces a non-degenerate symmetric bilinear form on each of the irreducible DK modules.

On the other hand, $D(V, B)$ may be realized inside the tensor product of two copies of the Clifford algebra, $C(V, B)$, as the subalgebra generated by the elements of the form, $\frac{1}{2}(1 \otimes u + u \otimes 1)$, with $u \in V$. Therefore, each pair of representations of $C(V, B)$ give a representation of $D(V, B)$. In particular, we study the representations arising from the tensor product of two copies of the fundamental 2^{q+1}-dimensional $C(q+2, q)$-module, $W(q+2, q)$ ($q = 0$, 1, and 2). This yields a faithful representation of $D(q+2, q)$ in the space, $\boldsymbol{R}^{2^{q+1}\times 2^{q+1}}$, of real $2^{q+1} \times 2^{q+1}$ matrices. This representation is completely reducible and yields, with multiplicity one, *all* the irreducible representations of $D(q+2, q)$. The complete reducibility follows from geometrical reasons. In fact, $\boldsymbol{R}^{2^{q+1}\times 2^{q+1}}$ has an orthogonal structure, $(\cdot, \cdot)_q$, induced by the spin group, $Spin(q+2, q)$, in the tensor product $W(q+2, q) \otimes W(q+2, q)$. It is given by

$$(\xi, \zeta)_q = \mathrm{Tr}(\xi (b_q \zeta b_q{}^t)^t) \qquad \mathrm{sgn}(\cdot, \cdot)_q = \begin{cases} (4, 0) & \text{if } q = 0 \\ (8, 8) & \text{if } q = 1 \\ (32, 32) & \text{if } q = 2 \end{cases}$$

b_q being an invertible matrix defined in a natural manner for each case (we have included the details in the appendix). In particular, the overall geometric structure induced in $\boldsymbol{R}^{2^{2(q+1)}}$ is the same as the one induced in $\wedge(\boldsymbol{R}^{2(q+1)})$ by B. Furthermore, it is shown that for each generator $a \in D(q+2, q)$, there is a scalar, $\lambda_a \in \boldsymbol{R}$, such that, for all ξ, and $\zeta \in \boldsymbol{R}^{2^{q+1}\times 2^{q+1}}$,

$$(a\xi, \zeta)_q + \lambda_a (\xi, a\zeta)_q = 0.$$

This makes the orthogonal complements of the invariant subspaces, invariant. Another property of this representation is that it leaves the subspaces, $S^2(\boldsymbol{R}^{2^{q+1}})$ and $A^2(\boldsymbol{R}^{2^{q+1}})$, of symmetric and skew-symmetric matrices, respectively, invariant. For $q = 0$ they are both irreducible and exhaust the list of simple $D(2, 0)$-modules. For $q = 1$, only the subspace of symmetric matrices is irreducible. The skew-symmetric matrices split into the direct sum of the one-dimensional trivial module—defined by the common intersection of the kernels of (the operators that represent) the generators of $D(3,1)$—and its orthogonal complement in $A^2(\mathbf{R}^4)$. Moreover, $(\cdot,\cdot)_1$ restricts to $A^2(\mathbf{R}^4)$ with signature $(4,2)$, and it is negative definite on the one-dimensional trivial submodule. Finally, for $q=2$, both $S^2(\mathbf{R}^8)$ and $A^2(\mathbf{R}^8)$ are reducible. This time the trivial one-dimensional submodule occurs inside $S^2(\mathbf{R}^8)$. Its orthogonal complement in $S^2(\mathbf{R}^8)$ is the 35-dimensional simple module. The restriction of $(\cdot,\cdot)_2$ to $S^2(\mathbf{R}^8)$ has signature $(16,20)$ and it is negative definite on the one-dimensional trivial submodule. On the other hand, $A^2(\mathbf{R}^8)$ splits into two simple modules of dimensions 7 and 21, and the form $(\cdot,\cdot)_2$ restricts to them with signatures $(4,3)$, and $(12,9)$, respectively.

Let us briefly explain here how the orthogonal form on $\mathbf{R}^{2^{q+1}\times 2^{q+1}}$ is obtained from $Spin(q+2,q)$. First, there is a natural identification of $W(q+2,q)$ with $\mathbf{C}^{2^q}$. In fact, $C(q+2,q)$ contains an element with square -1, γ_{2q+3}, uniquely and invariantly defined up to a sign. Therefore, in every non-trivial representation of $C(q+2,q)$, γ_{2q+3} defines a complex structure on the corresponding representation module. This was observed in a previous work by one of us and Sternberg [8], while studying the Clifford algebras $C(q+2,q)$ in connection with the connectivity properties of the spin groups and their relevance for conformal supersymmetry. Now, the spin group, $Spin(q+2,q)$ endows $W(q+2,q)\simeq\mathbf{C}^{2^q}$ with a specific geometric structure, according to the following scheme:

$$Spin(q+2,q)_e \simeq \begin{cases} U(1) & \text{if } q=0 \\ SL(2,\mathbf{C}) & \text{if } q=1 \\ SU(2,2) & \text{if } q=2 \end{cases}$$

$Spin(q+2,q)_e$ being the identity component of $Spin(q+2,q)$. These structures induce specific geometries on the underlying real spaces $W(q+2,q)\simeq\mathbf{R}^{2^{q+1}}$, according to

$$W(q+2,q) \simeq \begin{cases} SO(2)\text{-module} & \text{if } q=0 \\ Sp(4,\mathbf{R})\text{-module} & \text{if } q=1 \\ *\begin{cases} Sp(8,\mathbf{R})\text{-module} \\ \quad\text{or} \\ O(4,4)\text{-module} \end{cases} & \text{if } q=2. \end{cases}$$

The first case follows since $U(1)\simeq SO(2)$. For the second, $SL(2,\mathbf{C})\simeq Sp(2,\mathbf{C})$ and both real and imaginary parts of a complex symplectic form define real symplectic structures on the underlying real space; thus, $Sp(2,\mathbf{C})\hookrightarrow Sp(4,\mathbf{R})$. Finally, either $SU(2,2)\hookrightarrow O(4,4)$ or, $SU(2,2)\hookrightarrow Sp(8,\mathbf{R})$, depending on whether the real or the imaginary part of the Hermitian structure (of signature $(2,2)$) in $W(4,2)\simeq\mathbf{C}^4$, is used to define a real bilinear form on $W(4,2)\simeq\mathbf{R}^8$. These two inclusions are physically relevant: $SU(2,2)\hookrightarrow Sp(8,\mathbf{R})$ was used by Sternberg [9] to obtain a conformally invariant notion of charge conjugation, and $SU(2,2)\hookrightarrow O(4,4)$ is best when dealing with conformally invariant solutions of Maxwell's equations as suggested by Howe [4]. For us, and as far as the simple $D(q+2,q)$-modules are concerned, the remarkable property is that the induced geometric structure on $W(q+2,q)\otimes W(q+2,q)$ is unambiguous for the $q=2$ case, regardless of whether $W(4,2)$ is taken with its $O(4,4)$, or with its $Sp(8,\mathbf{R})$ structure. Whence

$$W(q+2,q)\otimes_{\mathbb{R}} W(q+2,q) \simeq \begin{cases} O(4)\text{-module} & \text{if } q=0 \\ O(8,8)\text{-module} & \text{if } q=1 \\ O(32,32)\text{-module} & \text{if } q=2. \end{cases}$$

This follows from the way in which orthogonal and symplectic modules tensor with each other; the general result has been included in the appendix for reference.

2. Definition of the DK algebras

Convention. We shall adhere to the convention that all algebras have unit element, and we shall denote the unit of the algebra by 1_A. Moreover, a linear map, $\phi: A \to B$, between two $\mathbb{R}$-algebras is an algebra morphism only if the pair of conditions, $\phi(a_1a_2) = \phi(a_1)\phi(a_2)$ for all a_1, a_2 in A and $\phi(1_A) = 1_B$ are satisfied. In particular, for any representation, $\beta: A \to \operatorname{End} U$, we have, $\beta(1_A) = \mathrm{id}_U$.

Let V be a finite-dimensional vector space over the real or complex numbers, and let B be a non-degenerate, symmetric, bilinear form on V. Let A be any associative algebra and let $\phi: V \to A$ be a linear map. We shall say that ϕ has the DK *property* (or that ϕ is a DK *map*) if

$$\phi(u)\phi(v)\phi(u) = B(u, v)\phi(u) \tag{1a}$$

for all u, and v in V. Equivalently,

$$\phi(u)\phi(v)\phi(w) + \phi(w)\phi(v)\phi(u) = B(u, v)\phi(w) + B(w, v)\phi(u). \tag{1b}$$

Definition 1. A DK algebra for the pair (V, B) is an associative algebra, $D(V, B)$, equipped with a DK map $\iota: V \to D(V, B)$ with the following universal property: for any associative algebra A and any DK map $\phi: V \to A$ there exists a unique algebra morphism $\Phi: D(V, B) \to A$ such that $\Phi \circ \iota = \phi$.

Posed this way, a DK algebra for the pair (V, B) exists and is unique up to isomorphism. The uniqueness follows as in the solution to any universal problem. The existence is proved by setting $D(V, B)$ equal to the tensor algebra of V, $\bigotimes(V)$, modulo the ideal, I, generated by the elements of the form $u \otimes v \otimes u - B(u, v)u$. The DK map $\iota: V \to D(V, B)$ is the composition of the natural injection $V \hookrightarrow \bigotimes(V)$ followed by the projection $\bigotimes(V) \to \bigotimes(V)/I$. Moreover, ι thus defined is injective and therefore V may (and shall) be identified with a subspace of $D(V, B)$. The verification that these prescriptions yield the universal property of the definition above is straightforward; it will therefore be omitted.

Proposition 2. The algebra $D(V, B)$ thus obtained is finite dimensional and $\dim D(V, B) = \binom{2n+1}{n}$ where n is the dimension of V.

Proof. Since $D(V, B)$ is a homomorphic image of $\bigotimes(V)$, it inherits a $\mathbb{Z}$-gradation. Let $D^k(V, B)$ be $\bigotimes^k(V)$ mod I. Thus, $D(V, B) = \bigoplus_{k\geqslant 0} D^k((V, B)$. Let $\{0\} \subset F^0(V, B) \subset \cdots F^k(V, B) \subset F^{k+1}(V, B) \subset \cdots$, be the filtration associated with the $\mathbb{Z}$-gradation; i.e. $F^k(V, B) = \bigoplus_{j\leqslant k} D^j(V, B)$. Note that, for all u, v, and w in V, we have

$$uvw \equiv -wvu \pmod{F^1(V, B)} \tag{2a}$$

$$u^3 = B(u, u)u \pmod{F^1(V, B)} \tag{2b}$$

$$u \otimes v \otimes u \equiv 0 \pmod{F^1(V, B)}. \tag{2c}$$

The first and third congruences become strict equalities when $B(u, v) = B(w, v) = 0$, and when $B(u, v) = 0$, respectively. These relations can be used to obtain a linearly independent subset of vector space generators of $D(V, B)$: Let $\{e_\mu\}$ be some orthonormal basis of V, and let $e_{\mu_1}e_{\mu_2}\cdots e_{\mu_k} \in D(V, B)$ stand for $e_{\mu_1} \otimes e_{\mu_2} \otimes \cdots \otimes e_{\mu_k}$ mod I; let $\bar{K}$ be the set of all sequences, $\mu = (\mu_1, \mu_2, \ldots, \mu_k)$, of length k, with, $\mu_i \in \{1, 2, \ldots, n\}$. Then, the subset $\{e_{\mu_1} e_{\mu_2} \cdots e_{\mu_k} \mid (\mu_1, \mu_2, \ldots, \mu_k) \in \bar{K}\}$, generates $D^k(V, B)$ mod $F^{k-1}(V, B)$. Moreover, it follows from (2a) that, for each integer, $1 \leqslant j \leqslant k-2$,

$$e_{\mu_1} e_{\mu_2} \cdots e_{\mu_j} e_{\mu_{j+1}} e_{\mu_{j+2}} \cdots e_{\mu_k} \equiv e_{\mu_1} e_{\mu_2} \cdots e_{\mu_{j+2}} e_{\mu_{j+1}} e_{\mu_j} \cdots e_{\mu_k} \pmod{F^{k-2}(V, B)}. \tag{3}$$

Note that if $\mu_j = \mu_{j+2} = \mu_{j+1}$ (respectively, $\mu_j = \mu_{j+2} \neq \mu_{j+1}$), (2$b$) (respectively, (2$c$)) implies that

$$e_{\mu_1} e_{\mu_2} \cdots e_{\mu_j} e_{\mu_{j+1}} e_{\mu_{j+2}} \cdots e_{\mu_k} \equiv 0 \pmod{F^{k-2}(V, B)}. \tag{4}$$

Thus, it may be assumed that $\mu_j < \mu_{j+2}$, and, therefore, if $k = 2s$, either $e_{\mu_1} \cdots e_{\mu_k} = 0$ or

$$e_{\mu_1} e_{\mu_2} \cdots e_{\mu_k} \equiv e_{i_1} e_{j_1} e_{i_2} e_{j_2} \cdots e_{i_s} e_{j_s} \pmod{F^{k-2}(V, B)} \tag{5a}$$

for one and only one pair of sequences $(i_1, \ldots, i_s)$ and $(j_1, \ldots, j_s)$ satisfying, $1 \leqslant i_1 < i_2 < \cdots < i_s \leqslant n$, and $1 \leqslant j_1 < j_2 < \cdots < j_s \leqslant n$. Similarly, if $k = 2s+1$, with $s \geqslant 1$,

$$e_{\mu_1} e_{\mu_2} \cdots e_{\mu_k} \equiv e_{i_1} e_{j_1} e_{i_2} e_{j_2} \cdots e_{i_s} e_{j_s} e_{i_{s+1}} \pmod{F^{k-2}(V, B)} \tag{5b}$$

with $(i_1, \ldots, i_{s+1})$ and $(j_1, \ldots, j_s)$ satisfying, $1 \leqslant i_1 < i_2 < \cdots < i_s < i_{s+1} \leqslant n$, and $1 \leqslant j_1 < j_2 < \cdots < j_s \leqslant n$. It follows that for $k \leqslant 2n$,

$$\dim D^k(V, B) = \begin{cases} \binom{n}{s}^2 & \text{if } k = 2s \\ \binom{n}{s+1}\binom{n}{s} & \text{if } k = 2s+1. \end{cases} \tag{6}$$

On the other hand, $\dim D^k(V, B) = 0$ if $k > 2n$. This follows from (4) and the fact that if $k > 2n$, there is at least one pair of repeated i or j indices when the μs in $e_{\mu_1} e_{\mu_2} \ldots e_{\mu_k}$ are relabelled so as to have $e_{\mu_1} e_{\mu_2} \ldots = e_{i_1} e_{j_1} e_{i_2} e_{j_2} \ldots$. In particular, $D(V, B)$ is finite dimensional and

$$\begin{aligned} \dim D(V, B) &= 1 + n + n^2 + n\binom{n}{2} + \binom{n}{2}^2 + \cdots + \binom{n}{n}\binom{n}{n-1} + 1 \\ &= \sum_0^n \left(\binom{n}{k} + \binom{n}{k-1}\right)\binom{n}{k} = \sum_0^n \binom{n+1}{k}\binom{n}{k} \\ &= \text{coefficient of } t^n \text{ in } (1+t)^n (1+t)^{n+1} = \binom{2n+1}{n} \end{aligned} \qquad \square$$

From the algebraic-theoretic point of view, the next step is the classification of the irreducible $DK(V, B)$-modules. As was pointed out to us by Professor S Sternberg, most of the arguments in the proof of the following theorem were already present in Kemmer's paper [7]. The modern-language proof given here is due to Sternberg.

Theorem 3. Let $D(V, B)$ be the DK algebra for the pair (V, B). Let $\wedge(V) = \oplus_{k \geqslant 0} \wedge^k(V)$ be the exterior algebra of V and for each $k = 0, 1, \ldots, n = \dim V$, set, $M_k = \wedge^k(V) \oplus \wedge^{k+1}(V)$.

(i) M_k has the structure of a $D(V, B)$-module (of dimension $\binom{n+1}{k+1}$).

(ii) The modules M_k, and M_{n-k-1} are isomorphic for $k = 0, 1, \ldots, n-1$, and M_n is a trivial one-dimensional module.

(iii) If n is even, $M_0, M_1, \ldots, M_{n/2}$, are all irreducible, and together with M_n, give the complete list of simple $D(V, B)$-modules.

(iv) If n is odd, $M_0, M_1, \ldots, M_{(n-3)/2}$, are all irreducible; $M_{(n-1)/2}$ splits into the direct sum of two (non-equivalent) irreducibles, each of dimension $\binom{n}{(n-1)/2}$. These, together with M_n, form the complete list of simple $D(V, B)$-modules.

Proof. We shall fix some notation and conventions. First, the bilinear form B on V, induces a bilinear form on $\wedge(V)$—also denoted by B—under which $B(\wedge^j(V), \wedge^k(V)) = 0$, whenever $j \neq k$. For each $\eta \in \wedge^k(V)$, $e(\eta)$ is the map $\wedge(V) \to \wedge(V)$ of exterior multiplication by η from the left; $i(\eta)$ is the adjoint of $e(\eta)$ with respect to B. If the signature of B is (p, q), our choice of orientation in V is given by $vol \in \wedge^n(V)$, with $B(vol, vol) = (-1)^q$. Finally, Hodge's star operator is defined by $*\omega = \mathrm{i}(\omega)vol$. In particular, $B(*\omega, *\omega) = (-1)^q B(\omega, \omega)$, for all $\omega \in \wedge(V)$, and $(* \circ *)|_{\wedge^k(V)} = (-1)^{q+k(n-k)} \mathrm{id}_{\wedge^k(V)}$.

(i) For each $u \in V$, let, $e_k(u): \wedge^k(V) \to \wedge^{k+1}(V)$, (respectively, $i_k(u): \wedge^k(V) \to \wedge^{k-1}(V)$), be the restriction to $\wedge^k(V)$ of the exterior (respectively, interior) multiplication by u. Define $\beta_k(u): \wedge^k(V) \oplus \wedge^{k+1}(V) \to \wedge^k(V) \oplus \wedge^{k+1}(V)$ by

$$\beta_k(u) = \begin{pmatrix} 0 & i_{k+1}(u) \\ e_k(u) & 0 \end{pmatrix}. \tag{7}$$

Since, $i_{k+1}(u)e_k(v) = B(u, v)\mathrm{id} - e_k(v)i_{k+1}(u)$, $e_{k+1}e_k(u) = 0$, and $i_{k+1}(u)i_k(u) = 0$, one easily verifies that

$$\beta_k(u)\beta_k(v)\beta_k(u) = B(u, v)\beta_k(u)$$

and the universal property of definition 1, extends β_k to a representation of $D(V, B)$. This proves (i).

(ii) The fact that M_n is a one-dimensional module follows trivially. On the other hand, the equivalence between M_k, and M_{n-k-1} is given by Hodge's star operator. More precisely, we claim that $* \circ \beta_k(u) = (-1)^k \beta_{n-k-1}(u) \circ *$. To prove this one must show that $i_{n-k}(u) \circ * = (-1)^k * \circ e_k(u)$, and $* \circ i_{k+1}(u) = e_{n-k-1}(u) \circ *$. However, the second equation follows from the first and the appropriate formulae for $* \circ *$. Now, for any $\omega \in \wedge^k(V)$, we have

$$i_{n-k}(u) * \omega = i_{n-k}(u)i(\omega)vol = i(\omega \wedge u)vol = (-1)^k i(u \wedge \omega)vol = (-1)^k * (u \wedge \omega)$$

and our claim follows.

(iii) We restrict ourselves to $0 < k \leqslant n/2$. Observe that

$$\begin{pmatrix} 0 & i_{k+1}(u) \\ e_k(u) & 0 \end{pmatrix}\begin{pmatrix} 0 & i_{k+1}(v) \\ e_k(v) & 0 \end{pmatrix} = \begin{pmatrix} i_{k+1}(u)e_k(v) & 0 \\ 0 & e_k(u)i_{k+1}(v) \end{pmatrix}$$
$$= \begin{pmatrix} I - e_k(v)i_{k+1}(u) & 0 \\ 0 & e_k(u)i_{k+1}(v) \end{pmatrix}.$$

Now, $e_k(v)i_{k+1}(u)$ is the derivation induced on $\wedge(V)$ by the rank-one element, $v \otimes u \in \mathrm{gl}(V)$ (which, under the identification of V and V^* via B, is defined on $\wedge^1(V) \simeq V$ by $(u \otimes v)w = B(u, w)v$) and such elements span the whole algebra $\mathbf{gl}(V)$. Now, $\mathbf{gl}(V)$ acts irreducibly on $\wedge^k(V)$ and the representations of $\mathbf{gl}(V)$ on $\wedge^k(V)$, and $\wedge^{k+1}(V)$ are all inequivalent except when, $k = \frac{1}{2}(n-1)$ (which does not occur, however, when n is even). Hence, the products $\beta_k(u)\beta_k(v)$ generate all of $\mathrm{End}(\wedge^k(V))/(I) \oplus \mathrm{End}(\wedge^{k+1}(V))$. This implies that if W is an invariant subspace of M_k, then $W = W \cap \wedge^k(V) \oplus W \cap \wedge^{k+1}(V)$ and that each of these intersections is either $\{0\}$ or the whole space. On the other hand, $\beta_k(u)$ is a non-zero map which interchanges the two components. Hence, either $W = \{0\}$ or $W = M_k$; i.e. the representation is irreducible. Furthermore, for $k = 0, 1, \ldots, n/2$, the M_k s are clearly inequivalent (in fact they even have different dimensions). Finally, to prove that these M_ks, together with the trivial module, M_n, are all the irreducibles, it is enough to show that the sum of the squares of the dimensions add up to the dimension of $D(V, B)$. Now, the sum of squares is

$$1+\sum_{1}^{n/2}\binom{n+1}{k+1}^2=\frac{1}{2}\sum_{0}^{n}\binom{n+1}{k}^2=\tfrac{1}{2}\text{ coefficient of } t^{n+1} \text{ in } (1+t)^{n+1}(1+t)^{n+1}$$
$$=\frac{1}{2}\binom{2n+2}{n+1}=\binom{2n+1}{n}=\dim D(V,B)$$

which completes the proof.

(iv) When $k = 0, 1, \ldots, (n-3)/2$, the irreducibility of the M_k s follows as in (iii). When $k = (n-1)/2$, the dimensions of $\wedge^k(V)$, and $\wedge^{k+1}(V)$ are the same, and Hodge's star operator interchanges these two. Thus, $*$ leaves $M_{(n-1)/2}$ invariant. Its splitting into the direct sum of two non-equivalent irreducibles of dimension $\binom{n+1}{(n-1)/2}$ each corresponds to the $*$-decomposition into self-dual, and anti-self-dual pieces. Finally, the fact that the sum of the squares of the dimensions of the non-decomposables is the dimension of $D(V, B)$ follows by a simple computation as in (iii). □

A consequence of the explicit realization of the various irreducible DK modules is the following: each M_k comes equipped with a non-degenerate symmetric bilinear form; namely the direct sum of the restrictions of B to $\wedge^k(V)$, and $\wedge^{k+1}(V)$, respectively.

Proposition 4. Let $D(V, B)$ be the DK algebra for the pair (V, B). Let $\wedge(V) = \oplus_{k\geqslant 0}\wedge^k(V)$ be the exterior algebra of V, and let B_k be the restriction to $\wedge^k(V)$ of the bilinear form induced on $\wedge(V)$ by B. Let $M_k = \wedge^k(V) \oplus \wedge^{k+1}(V)$ and let $\beta_k\colon D(V, B) \to \operatorname{End} M_k$ be the representation described in theorem 3. Then,

(i) β_k is orthogonal; i.e. for each $u \in V$, and all $\omega, \omega' \in \wedge^k(V)$, and $\sigma, \sigma' \in \wedge^{k+1}(V)$,

$$(B_k\oplus B_{k+1})(\beta_k(u)(\omega+\sigma),\omega'+\sigma')-(B_k\oplus B_{k+1})(\omega+\sigma,\beta_k(u)(\omega'+\sigma'))=0.$$

(ii) Let (p, q) be the signature of B on V. The signature of B_k is given by

$$\operatorname{sgn} B_k=\left(\sum_{r+2s=k}\binom{p}{r}\binom{q}{2s},\ \sum_{r+2s+1=k}\binom{p}{r}\binom{q}{2s+1}\right).$$

Proof. (i) From the previous theorem,

$$\beta_k(u)(\omega+\sigma)=i_{k+1}(u)\sigma+e_k(u)\omega.$$

Now, $B(i_{k+1}(u)\sigma, \sigma') = 0$, and $B(e_k(u)\omega, \omega') = 0$, due to the perpendicularity of $\wedge^k(V)$, and $\wedge^{k+1}(V)$ with respect to B. Since, the operators $e(u)$, and $i(u)$ in $\wedge(V)$, are adjoint to each other with respect to B, we have

$$B_{k+1}(e_k(u)\omega,\sigma')=B_k(\omega,i_{k+1}(u)\sigma') \qquad \text{and} \qquad B_k(i_{k+1}(u)\sigma,\omega')=B_{k+1}(\sigma,e_k(u)\omega')$$

from which assertion (i) follows.

(ii) Let $\{e_\mu \mid 1 \leqslant \mu \leqslant n = p+q\}$ be an orthonormal basis of V. Assume $B(e_\mu, e_\mu) = 1$ (respectively, $B(e_\mu, e_\mu) = -1$), if $1 \leqslant \mu \leqslant p$, (respectively, if $p+1 \leqslant \mu \leqslant n$). Then, $\{e_{\mu_1} \wedge \cdots \wedge e_{\mu_k} \mid 1 \leqslant \mu_1 < \cdots < \mu_k \leqslant n\}$ is an orthonormal basis of $\wedge^k(V)$ and

$$B_k(e_{\mu_1}\wedge\cdots\wedge e_{\mu_k}, e_{\mu_1}\wedge\cdots\wedge e_{\mu_k})=B(e_{\mu_1},e_{\mu_1})\cdots B(e_{\mu_k},e_{\mu_k}).$$

Now, the right-hand side is $+1$, only if the number of e_{μ_j}s with $B(e_{\mu_j}, e_{\mu_j}) = -1$ is even. Thus the various contributions to the signature of B_k can be counted by first decomposing $\binom{n}{k}$ as the sum, $\sum\binom{p}{i}\binom{q}{j}$ (running through all i, and j, with $i + j = k$), and then separating those with even j. □

Remark 5. Let us denote by (p_k, q_k), the signature of B_k as given in the proposition above. Due to the identification of $O(p_k, q_k)$, with $O(q_k, p_k)$, the actual signature of the induced bilinear form on M_k might come out with any of four possibilities: $(p_k + p_{k+1}, q_k + q_{k+1})$, $(p_k + q_{k+1}, q_k + p_{k+1})$, $(q_k + p_{k+1}, p_k + q_{k+1})$ or $(q_k + q_{k+1}, p_k + p_{k+1})$. It turns out that in the realizations of the DK-modules we study in section 4, the signatures on the various M_ks appear in an alternating fashion according to table 1 (see section 5).

Table 1. Signatures of the orthogonal forms on the non-decomposable DK-modules, M_k, induced by an orthogonal form of signature $(q+2, q)$ on V ($q = 0$, 1 and 2).

	$q=0$	$q=1$	$q=2$
M_0	$(p_0+p_1, q_0+q_1) = (3,0)$	$(p_0+p_1, q_0+q_1) = (4,1)$	$(q_0+p_1, p_0+q_1) = (4,3)$
M_1		$(q_1+p_2, p_1+q_2) = (4,6)$	$(p_1+q_2, q_1+p_2) = (12,9)$
M_2			$(q_2+p_3, p_2+q_3) = (16,19)$
M_{q+2}	$(p_{q+2}, q_{q+2}) = (1,0)$	$(p_{q+2}, q_{q+2}) = (0,1)$	$(q_{q+2}, p_{q+2}) = (0,1)$

Remark 6. Note that the sum of the dimensions of all the irreducible DK-modules is exactly the dimension of the exterior algebra, $\wedge(V)$. Therefore, the direct sum of all the irreducible representations gives a representation $\beta\colon D(V,B) \to \mathrm{End}(\wedge(V))$. When $n = \dim V = 2m$, we have

$$\beta\colon D(V,B) \to \mathrm{End}(\boldsymbol{R}^{2^{2m}}) \simeq \mathrm{End}(\boldsymbol{R}^{2^m \times 2^m}).$$

In other words, one may realize in the space of $2^m \times 2^m$ matrices, the completely reducible representation uniquely characterized by having each irreducible of $D(V,B)$ with multiplicity one in its decomposition. Similarly, when $n = 2m+1$, one may realize this unique representation in $\boldsymbol{R}^{2^m \times 2^m} \oplus \boldsymbol{R}^{2^m \times 2^m}$.

Remark 7. Since the generators, $\{e_\mu\}$, of the DK algebra, $D(V,B)$, satisfy the relations (2*b*), their images, $\beta_\mu = \beta(e_\mu)$, under any representation, $\beta\colon D(p,q) \to \mathrm{End}\, U$, have either $X^3 - X$, or $X^3 + X$, as minimal polynomial. In particular, $\lambda = 0$ is an eigenvalue for all the β_μs. Hence, $\mathrm{Ker}\,\beta_\mu \neq \{0\}$. If it furthermore happens that $\dim \bigcap_\mu \mathrm{Ker}\,\beta_\mu = 1$, the irreducible, one-dimensional, trivial representation occurs with multiplicity one in U. Again, this will be the case in the realizations given in section 4.

3. Representations of DK algebras from Clifford algebras

We shall now show how to find representations of the DK algebras, in terms of representations of Clifford algebras. Certain familiarity with the latter will be assumed (e.g., see [8], and references therein). The notation is the following: $C(V,B)$ is the Clifford algebra associated with the pair (V,B), with (V,B) as in the previous section. In concrete examples, explicit reference shall be made to the signature and we shall write $C(p,q)$, $D(p,q)$, etc, instead of $C(V,B)$, $D(V,B)$, etc, whenever $\mathrm{sgn}\, B = (p,q)$.

Remark 8. The basic observation on which the following developments are based is this: in the tensor product algebra $C(V,B) \otimes C(V,B)$ (with product defined by $(a \otimes b)(a' \otimes b') = aa' \otimes bb'$), the subalgebra generated by the elements

$$\beta(u) = \tfrac{1}{2}(u \otimes 1 + 1 \otimes u) \qquad u \in V \subset C(V,B) \tag{8}$$

satisfies the DK defining relations (1*a*). This follows at once from the Clifford algebra relations, $uv + vu = 2B(u,v)1$.

Proposition 9. Let $\gamma_1: C(V, B) \to \mathrm{End}(W_1)$ and $\gamma_2: C(V, B) \to \mathrm{End}(W_2)$ be any two representations of the Clifford algebra $C(V, B)$. Then, the map $\beta: D(V, B) \to \mathrm{End}(W_1 \otimes W_2)$, defined by

$$\beta(\cdot) = \tfrac{1}{2}(\gamma_1(\cdot) \otimes \mathrm{id}_{W_2} + \mathrm{id}_{W_1} \otimes \gamma_2(\cdot))$$

yields a representation of $D(V, B)$ on $W_1 \otimes W_2$.

Proof. This is a direct calculation from the definitions. It suffices to check that (1a) is satisfied by β when applied to any decomposable vector $w_1 \otimes w_2 \in W_1 \otimes W_2$. Thus, for any pair of vectors, u, and $v \in V$, we have

$$\begin{aligned}\beta(u) \circ \beta(v) \circ \beta(u)(w_1 \otimes w_2) = \tfrac{1}{8}(&\gamma_1(u)\gamma_1(v)\gamma_1(u)w_1 \otimes w_2 + \gamma_1(v)\gamma_1(u)w_1 \otimes \gamma_2(u)w_2 \\ &+ \gamma_1(u)\gamma_1(v)w_1 \otimes \gamma_2(u)w_2 + \gamma_1(v)w_1 \otimes \gamma_2(u)^2 w_2 \\ &+ \gamma_1(u)^2 w_1 \otimes \gamma_2(v)w_2 + \gamma_1(u)w_1 \otimes \gamma_2(u)\gamma_2(v)w_2 \\ &+ \gamma_1(u)w_1 \otimes \gamma_2(v)\gamma_2(u)w_2 + w_1 \otimes \gamma_2(u)\gamma_2(v)\gamma_2(u)w_2).\end{aligned}$$

Since the γ_is are representations of $C(V, B)$, they satisfy $\gamma_i(u)\gamma_i(v) + \gamma_i(v)\gamma_i(u) = 2B(u, v)\,\mathrm{id}_{W_i}$; $i = 1, 2$. Using these relations, the right-hand side above simplifies to $B(u, v)\beta(u)(w_1 \otimes w_2)$. □

Remark 10. (Kostant) In particular, $C(V, B)$ has the structure of a $C(V, B) \otimes C(V, B)$-module, by letting, for all a, b, and $\omega \in C(V, B)$, $(a \otimes b) \cdot c = ax\alpha(b)$; α being the principal anti-automorphism of $C(V, B)$ which is the identity on $V \subset C(V, B)$. Hence, for all $u \in V$, $(u \otimes 1 + 1 \otimes u) \cdot \omega = u\omega + \omega u$. Now, under the vector space identification, $C(V, B) \simeq \wedge(V)$, it is well known (and easy to prove) that, for all $\omega \in \wedge^k(V)$, $u\omega + \omega u = 2e_k(u)\omega$, if k is even, whereas $u\omega + \omega u = 2i_k(u)\omega$, if k is odd. Therefore, *the $D(V, B)$-modules, $\wedge^k(V) \oplus \wedge^{k+1}(V)$ of theorem 3 are the result of breaking up $C(V, B) \simeq \wedge(V)$ into $D(V, B)$-irreducibles; $D(V, B) \subset C(V, B) \otimes C(V, B)$.* (We are indebted to S Sternberg for communicating to us this argument of B Kostant, and to B Kostant himself for letting us include it here.)

Now, in order to realize explicitly the $D(V, B)$-irreducible representations in matrix spaces, we shall make use of the following trivial facts:

Proposition 11. Let the hypothesis and notation be as in proposition 9. Identify $W_1 \otimes W_2$ with the space of $\dim W_1 \times \dim W_2$-matrices, via

$$W_1 \otimes W_2 \ni X = \sum X_{ij} w_i^{(1)} \otimes w_j^{(2)} \longleftrightarrow X = (X_{ij})$$

for some choices, $\{w_i^{(1)}\}$, and $\{w_j^{(2)}\}$, of bases of W_1, and W_2. Then, the representation $\beta: D(V, B) \to \mathrm{End}(W_1 \otimes W_2)$, is given by

$$\beta(\cdot)(X) = \tfrac{1}{2}(\gamma_1(\cdot)X + X\gamma_2(\cdot)^t)$$

Proof. This again is a straightforward consequence of the definitions. Let $R: W_1 \to W_1$, and $S: W_2 \to W_2$ be linear maps, such that $R(w_j^{(1)}) = \sum_i R_{ij} w_i^{(1)}$ and $S(w_l^{(2)}) = \sum_i S_{kl} w_k^{(2)}$. Then,

$$R \otimes S\left(\sum X_{jl} w_j^{(1)} \otimes w_l^{(2)}\right) = \sum R_{ij} X_{jl} S_{kl} w_i^{(1)} \otimes w_k^{(2)} = \sum (RXS^t)_{ik} w_i^{(1)} \otimes w_k^{(2)}$$

where $(RXS^t)_{ik}$, denotes the (i, k)-entry of the product of the matrices, $R = (R_{ij})$, $X = (X_{jl})$, and the transpose of $S = (S_{kl})$. In particular, if X is identified with $X = (X_{jl})$, $R \otimes S(X)$ is identified with RXS^t, from which the statement follows. □

Table 2. Clifford modules $W(q+2,q)$ ($q = 0$, 1 and 2); their spin groups, their induced real bilinear forms, and suitable matrices, b_q, representing the bilinear forms induced on $W(q+2,q) \otimes_R W(q+2,q)$.

	$W(q+2,q)$	$Spin(q+2,q)_e$	$W(q+2,q)$ over R	$b_q \in R^{2^{q+1}\times 2^{q+1}}$
$q=0$	C	$U(1)$	R^2 with $O(2)$	$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}$
$q=1$	C^2	$SL(2,C)$	R^4 with $Sp(4,R)$	$\begin{pmatrix}J & 0\\ 0 & -J\end{pmatrix}$
$q=2$	C^4	$SU(2,2)$	R^8 with $\begin{cases} O(4,4) \\ Sp(4,R)\end{cases}$	$\begin{cases}\begin{pmatrix}K_2 & 0\\ 0 & K_2\end{pmatrix} \\ \begin{pmatrix}0 & -K_2\\ K_2 & 0\end{pmatrix}\end{cases}$

Proposition 12. Assume that $\gamma_1 = \gamma_2$. Then, the subspaces of symmetric and skew-symmetric matrices are invariant for the representation β.

Proof. Let the notation be as in the previous proposition. If $X^t = \pm X$, then

$$(\beta(u)X)^t = \tfrac{1}{2}(\gamma_1(u)X + X\gamma_2(u)^t)^t = \pm\tfrac{1}{2}(X\gamma_1(u)^t + \gamma_2(u)X)^t.$$

In particular, if $\gamma_1 = \gamma_2$, $(\beta(u)X)^t = \pm\beta(u)X$. $\square$

It will now be assumed that the symmetric bilinear form, B, of V has signature $(q+2, q)$, with $q = 0$, 1, or 2.

Recall that the simple Clifford modules $W(q+2,q)$, are naturally identified with $C^{2^q} \simeq R^{2^{q+1}}$ (see [8], and section 4 below). In particular, taken together with their natural geometric structures defined by the spin groups, $Spin(q+2,q)$, they induce natural geometric structures on the (real) tensor product spaces $W(q+2,q) \otimes W(q+2,q)$, according to the following scheme (cf appendix A):

$$W(q+2,q) \otimes_{\mathbb{R}} W(q+2,q) \simeq \begin{cases} O(4)\text{-module} & \text{if } q=0 \\ O(8,8)\text{-module} & \text{if } q=1 \\ O(32,32)\text{-module} & \text{if } q=2. \end{cases}$$

Furthermore, it follows from the proofs in the appendix that the orthogonal structure in the tensor product spaces depends only on the bilinear form with which the Clifford modules come equipped with, as follows: after identifying $W(q+2,q) \otimes_{\mathbb{R}} W(q+2,q)$ with the space of $2^{q+1} \times 2^{q+1}$-matrices, the orthogonal structure, $(\cdot,\cdot)_q$ is computed in terms of matrices ξ, and ζ, as

$$(\xi,\zeta)_q = \mathrm{Tr}(\xi(b_q \zeta {b_q}^t)^t) \tag{9}$$

where b_q is the matrix associated with the real geometric structure on $W(q+2,q)$. We summarize the relevant information in table 2, where

$$J = \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} \qquad \text{and} \qquad K_2 = \begin{pmatrix}1_2 & 0\\ 0 & -1_2\end{pmatrix} \qquad \text{with} \qquad 1_2 = \begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

We can now state the following result, which is a consequence of proposition 4, remark 10, and our computations in section 4 below (see also remark 5, and table 1).

Proposition 13. The geometry induced by $Spin(q+2,q)$ on $W(q+2,q) \otimes W(q+2,q)$ coincides with the geometry induced by B on $\wedge(\boldsymbol{R}^{2(q+1)})$, described in theorem 3.

Thus, the rest of our exposition (the next two sections) will focus on realizing explicitly the completely reducible DK-modules of remark 6 for the cases under consideration. We shall need only one technical point that we record in the following:

Proposition 14. Let $\beta: A \to \operatorname{End} U$ be a representation of some associative $\boldsymbol{R}$-algebra A. Let $(\cdot,\cdot)$ be a given geometric structure on U (i.e. an orthogonal, symplectic or Hermitian form). Suppose that for each $a \in A$, there exists some $\lambda_a \in \boldsymbol{R} - \{0\}$, such that

$$(\beta(a)\xi, \zeta) + \lambda_a(\xi, \beta(a)\zeta) = 0.$$

Then, the orthogonal subspace, $W^{\perp}$, of any invariant subspace, $W \subset U$, is invariant, too. In fact, the same conclusion is attained if this condition is satisfied only on a generating subset of A.

Proof. By definition, $W^{\perp} = \{\eta \in U \mid (\eta, \xi) = 0, \text{ for all } \xi \in W\}$. Let $a \in A$ be arbitrary and let $\xi \in W$. Since W is invariant, $\beta(a)\xi \in W$. Thus, $(\eta, \beta(a)\xi) = 0$, for all $\eta \in W^{\perp}$. By hypothesis, $(\beta(a)\eta, \xi) = -\lambda_a(\eta, \beta(a)\xi)$, which is zero. Therefore, $\beta(a)\eta \in W^{\perp}$.

Now, let $\{u_\mu \mid \mu \in \mathrm{M};\ \mathrm{M} \text{ finite}\}$ be a set of generators, and assume that $(\beta(u)\xi, \zeta) + \lambda_u(\xi, \beta(u)\zeta) = 0$ holds only for the us in the generating subset. Writing any $a \in A$ in the form, $a = u_1 u_2 \cdots u_n$, we obtain

$$(\beta(u_1 u_2 \cdots u_n)\eta, \xi) = (-1)^n \lambda_{u_1}\lambda_{u_2}\cdots\lambda_{u_n}(\eta, \beta(u_n)\cdots\beta(u_2)\beta(u_1)\xi)$$

for all η, and ξ in U. In particular, if ξ belongs to some invariant subspace, W, and η to its orthogonal complement, $W^{\perp}$, the left-hand side is zero. □

We shall now give an *ad hoc* criterion under which the hypotheses of proposition 14 are satisfied for the representation of β given in proposition 11 with $\gamma_1 = \gamma_2 = \gamma$. From (1), we have

$$\begin{aligned}(\beta(u)\xi, \zeta)_q + \lambda_u(\xi, \beta(u)\zeta)_q &= \operatorname{Tr}((\beta(u)\xi)(b_q\zeta b_q{}^t)^t) + \lambda_u \operatorname{Tr}(\xi(b_q(\beta(u)\zeta)b_q{}^t)^t) \\ &= \tfrac{1}{2}\{\operatorname{Tr}((\xi(b_q\zeta b_q{}^t)^t + \xi^t(b_q\zeta b_q{}^t)^t)(\gamma(u) + \lambda_u(b_q\gamma(u)b_q{}^{-1})^t))\}. \end{aligned} \tag{10}$$

Therefore, the result we seek is the following:

Criterion 15. If for each generator $u \in V$, the matrices $\gamma(u)$ have definite symmetry, then

$$R_q(u) = \gamma(u) + \lambda_u(b_q\gamma(u)b_q{}^{-1})^t$$

is identically zero and therefore the representation β of proposition 12 has the properties stated in proposition 14.

The truth of the assertion follows from (10) and the fact that b_q has definite symmetry. Therefore, the matrices $R_q(u)$ become zero for $\lambda_u = \pm 1$. To apply it, all what is needed is to look at the specific Dirac matrices of the representation γ of $C(q+2,q)$. This is done in the next section following the approach of [8].

4. Representations of DK algebras

Let $R^{q+2,q}$ be the vector space R^{2q+2}, equipped with the orthogonal form of signature $(q+2, q)$. Let $\{e_\mu\}$ be an orthonormal basis for $R^{q+2,q}$. We shall denote by γ_μ, and β_μ, the images of e_μ under the inclusions, $\gamma: R^{q+2,q} \to C(q+2,q) \simeq \mathrm{End}_R(C^{2^q})$, and $\beta: R^{q+2,q} \to D(q+2,q) \subset \mathrm{End}_R(R^{2^{q+1}\times 2^{q+1}})$, respectively. We recall that the fundamental Clifford module, $W(q+2,q) \simeq R^{2^{q+1}}$, is naturally identified with C^{2^q} because there is a Clifford algebra element,

$$\gamma_{2q+3} = \gamma_1\gamma_2\cdots\gamma_{2q+2} \tag{11}$$

satisfying

$$(\gamma_{2q+3})^2 = -1 \qquad \text{and} \qquad \gamma_{2q+3}\gamma_\mu = -\gamma_\mu\gamma_{2q+3}. \tag{12}$$

This element is invariantly defined, since it only changes by a sign under the action of the automorphism group of the algebra $C(q+2,q)$. Thus, γ_{2q+3} endows the real module $W(q+2,q)$ with a natural complex structure. Furthermore, the second set of relations in (12) imply that any complex representation of $C(q+2,q)$ in which the complex structure is defined by γ_{2q+3}, must represent the generators γ_μ by antilinear maps. In what follows we shall give to $W(q+2,q)$ this natural complex structure, and identify it with C^{2^q}. Moreover, we shall ocassionally think of the algebra of $2^{q+1}\times 2^{q+1}$ real matrices as $\mathrm{End}_R(W(q+2,q)) \simeq \mathrm{End}_R(C^{2^q})$, and decompose it as

$$\mathrm{End}_R(C^{2^q}) = \mathrm{End}_C(C^{2^q}) \oplus a_q \circ \mathrm{End}_C(C^{2^q}) \tag{13}$$

where $\mathrm{End}_C(C^{2^q})$ is identified with the algebra of $2^q\times 2^q$ complex matrices, and $a_q: C^{2^q} \to C^{2^q}$ is some fixed, invertible, antilinear map. In particular, we shall always have γ_μ in the second direct summand of this space.

4.1. Case $q = 0$

We identify $W(2,0) = R^2$ with C, in such a way that multiplication by $i = \sqrt{-1}$ is the real transformation,

$$J_2(w_1) = w_2 \qquad \text{and} \qquad J_2(w_2) = -w_1 \tag{14}$$

and

$$J_2 \longleftrightarrow \gamma_3 = \gamma_1\gamma_2.$$

Note that up to a complex constant, the only antilinear map $a_0: C \to C$ is complex conjugation, $\kappa: z \mapsto \bar{z}$. Thus, the setting is

$$\gamma_1 = \kappa \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in R^{2\times 2} \qquad \text{and} \qquad \gamma_2 = \kappa \circ i \leftrightarrow \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \in R^{2\times 2} \tag{15}$$

and both matrices have definite parity. Therefore, the representation of $D(2,0)$ they induce satifies criterion 15 hence, and, proposition 14. Also note that the representation space for

$D(2,0)$ is isomorphic to $\boldsymbol{C} \otimes_{\boldsymbol{R}} \boldsymbol{C}$, which in turn is isomorphic to the quaternionic space $\boldsymbol{H}$. One convenient identification is given by

$$1 \otimes 1 \leftrightarrow 1_H = \mathbf{1} \qquad i \otimes 1 \leftrightarrow i_H = \boldsymbol{i} \qquad 1 \otimes i \leftrightarrow j_H = \boldsymbol{j} \qquad i \otimes i \leftrightarrow k_H = \boldsymbol{k}. \tag{16}$$

Since $\gamma_1 : z \mapsto \bar{z}$ and $\gamma_2 : z \mapsto -i\bar{z}$, the prescription of proposition 9 gives

$$\beta_1(\mathbf{1}) = \mathbf{1} \qquad \beta_1(\boldsymbol{i}) = 0 \qquad \beta_1(\boldsymbol{j}) = 0 \qquad \beta_1(\boldsymbol{k}) = -\boldsymbol{k}$$

and

$$\beta_2(\mathbf{1}) = -\tfrac{1}{2}(\boldsymbol{i}+\boldsymbol{j}) \qquad \beta_2(\boldsymbol{i}) = -\tfrac{1}{2}(\mathbf{1}+\boldsymbol{k}) \qquad \beta_2(\boldsymbol{j}) = -\tfrac{1}{2}(\mathbf{1}+\boldsymbol{k})$$
$$\beta_2(\boldsymbol{k}) = -\tfrac{1}{2}(\boldsymbol{i}+\boldsymbol{j}).$$

These can be rewritten in terms of algebraic operations in H as follows: for all $q \in \boldsymbol{H}$,

$$\beta_1(q) = -\tfrac{1}{2}(\boldsymbol{i}q\boldsymbol{i} + \boldsymbol{j}q\boldsymbol{j}) \qquad \text{and} \qquad \beta_2(q) = \tfrac{1}{2}(\boldsymbol{i}q\boldsymbol{k} + \boldsymbol{k}q\boldsymbol{j}). \tag{17}$$

Also note that

$$\operatorname{Ker}\beta_1 = \operatorname{Span}_{\boldsymbol{R}}\{\boldsymbol{i},\boldsymbol{j}\} \qquad \text{and} \qquad \operatorname{Ker}\beta_2 = \operatorname{Span}_{\boldsymbol{R}}\{\mathbf{1}-\boldsymbol{k}, \boldsymbol{i}-\boldsymbol{j}\}$$

and therefore

$$\operatorname{Ker}\beta_1 \cap \operatorname{Ker}\beta_2 = \operatorname{Span}_{\boldsymbol{R}}\{\boldsymbol{i}-\boldsymbol{j}\}. \tag{18}$$

The semisimplicity of the representation becomes evident when we change the basis of $\boldsymbol{H}$ so as to obtain a matrix representation of the β_μs in which one of them is diagonal; say, β_2. Such a basis is $\{\boldsymbol{i}-\boldsymbol{j}, \mathbf{1}-\boldsymbol{k}, \mathbf{1}-\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k}, \mathbf{1}+\boldsymbol{i}+\boldsymbol{j}+\boldsymbol{k}\}$, and then

$$\beta_1 = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \qquad \text{and} \qquad \beta_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

It is therefore, clear that the matrices of the representation break up into the direct sum of two blocks: the trivial 1×1 block of the upper left corner, and the 3×3 block of the lower right. These yield precisely the two irreducible representations of $D(2,0)$. In particular, since $\dim D(2,0) = 10$, there must be a one-dimensional ideal in $D(2,0)$ that is represented by zero in the three-dimensional representation. It is easy to check that this ideal is generated by the element

$$\zeta = 1 - {\beta_1}^2 - {\beta_2}^2 + {\beta_1}^2{\beta_2}^2 = (1-{\beta_1}^2)(1-{\beta_2}^2).$$

Finally, when the representation space $W(2,0) \otimes W(2,0)$ is identified with the space of 2×2-matrices via

$$\begin{aligned} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} &\leftrightarrow e_1 \otimes e_1 \leftrightarrow \mathbf{1} \qquad & \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} &\leftrightarrow e_2 \otimes e_1 \leftrightarrow \boldsymbol{i} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} &\leftrightarrow e_1 \otimes e_2 \leftrightarrow \boldsymbol{j} \qquad & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} &\leftrightarrow e_2 \otimes e_2 \leftrightarrow \boldsymbol{k} \end{aligned} \tag{19}$$

the action of the β_μs is given by (see proposition 11)

$$\beta_1\begin{pmatrix} a & c \\ b & d \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & -d \end{pmatrix} \qquad \text{and} \qquad \beta_2\begin{pmatrix} a & c \\ b & d \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} b+c & a+d \\ a+d & b+c \end{pmatrix}. \tag{20}$$

When these expressions are used, the common kernel is defined by $a = d = 0$, and $b = -c$; i.e. the one-dimensional subspace of skew-symmetric matrices. Note that the three-dimensional space of symmetric matrices corresponds precisely to the subspace generated by $\{\mathbf{1} - \boldsymbol{k}, \mathbf{1} - \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}, \mathbf{1} + \boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}\}$, under the identifications above. Also note that this subspace is cyclic for the representation and hence, irreducible; e.g.

$$X = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \Longrightarrow \quad \beta_1(X) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \text{and} \quad \beta_2(X) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let us point out that the identifications (19) generalize to

$$\boldsymbol{R}^{2^{q+1}\times 2^{q+1}} \longleftrightarrow \boldsymbol{C}^{2^q} \otimes \boldsymbol{C}^{2^q} \longleftrightarrow \boldsymbol{H}^{2^q\times 2^q} \tag{21}$$

in the following way: the identification of $W(q+2, q) \simeq \boldsymbol{R}^{2^{q+1}}$ with $\boldsymbol{C}^{2^q}$ is given by

$$\boldsymbol{R}^{2^{q+1}} \ni (x_1, \ldots, x_{2^q}, y_1, \ldots, y_{2^q}) \longleftrightarrow (x_1 + \mathrm{i}y_1, \ldots, x_{2^q} + \mathrm{i}y_{2^q}) \in \boldsymbol{C}^{2^q}. \tag{22a}$$

Thus, it is assumed that γ_{2q+3} is represented by the matrix

$$J_{2^{q+1}} = \begin{pmatrix} 0 & -1_{2^q} \\ 1_{2^q} & 0 \end{pmatrix}. \tag{22b}$$

Then, $\boldsymbol{C}^{2^q} \otimes \boldsymbol{C}^{2^q}$ is identified with $\boldsymbol{H}^{2^q\times 2^q}$ as follows: let z_α be a column vector in $\boldsymbol{C}^{2^q}$ whose only non-zero component occurs in the αth row; let this be $x_\alpha + \mathrm{i}y_\alpha \in \boldsymbol{C}$. Then

$$\begin{aligned} \boldsymbol{C}^{2^q} \otimes \boldsymbol{C}^{2^q} \ni (z_\alpha \otimes z_\beta) &\longleftrightarrow ((x_\alpha + \mathrm{i}y_\alpha) \otimes (x_\beta + \mathrm{i}y_\beta)) \\ &= (x_\alpha x_\beta \mathbf{1} + y_\alpha x_\beta \boldsymbol{i} + x_\alpha y_\beta \boldsymbol{j} + y_\alpha y_\beta \boldsymbol{k}) \in \boldsymbol{H}^{2^q\times 2^q}. \end{aligned}$$

This yields the correspondence

$$\boldsymbol{R}^{2^{q+1}\times 2^{q+1}} \ni \begin{pmatrix} A & C \\ B & D \end{pmatrix} \longleftrightarrow (A\mathbf{1} + B\boldsymbol{i} + C\boldsymbol{j} + D\boldsymbol{k}) \in \boldsymbol{H}^{2^q\times 2^q} \tag{23}$$

where A, B, C and D are $2^q \times 2^q$ blocks.

4.2. Case $q = 1$

This time the Clifford module $W(3,1) \simeq \boldsymbol{R}^4$ is naturally equipped—via $Spin(3,1)_e \simeq SL(2,\boldsymbol{R})$—with symplectic geometry. In addition to the natural complex structure in $W(3,1)$ defined by γ_5, there is again a natural choice of the antilinear map a_1 (cf equation 13);i.e. namely, the (up to a constant) unique map, $*: \boldsymbol{C}^2 \to \boldsymbol{C}^2$, that intertwines over the real field the two inequivalent, two-dimensional, complex representations, $A \mapsto A$, and $A \mapsto (A^*)^{-1}$, of $SL(2,\boldsymbol{R})$, A^* being the conjugate transpose of A (see [8]). We recall that $*$ is given by

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \mapsto * \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} = \begin{pmatrix} \bar{z}_2 \\ -\bar{z}_1 \end{pmatrix}. \tag{24}$$

In particular, $* \circ * = -\,\mathrm{id}$, and one can directly verify that, $*A(*)^{-1} = (A^*)^{-1}$, for all $A \in SL(2, \mathbb{C})$. More generally, for any matrix $B \in \mathbb{C}^{2\times 2}$,

$$* B(*)^{-1} = (B^a)^*$$

with B^a the Cramer adjoint of B (i.e. the matrix defined by the relations $B^a B = (\det B)\,\mathrm{id} = BB^a$).

Now, to represent the Clifford algebra, we first use the one-to-one correspondence between the points in spacetime, $\mathbb{R}^{3,1}$, and the subset, $H(2) \subset \mathbb{C}^{2\times 2}$, of 2×2 Hermitian matrices given in terms of the Pauli matrices; namely

$$\mathbb{R}^{3,1} \ni X = (t, x, y, z) \leftrightarrow X = \begin{pmatrix} t+z & x-\mathrm{i}y \\ x+\mathrm{i}y & t-z \end{pmatrix} = t\sigma_0 + x\sigma_1 + y\sigma_2 + z\sigma_3 \in H(2)$$

where

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{25}$$

Observe that $B(X, X) = -\det X$. Also note that

$$(* \circ X)(* \circ X) = - * X(*)^{-1}X = -X^a X = -(\det X)1_2 = B(X, X)1_2$$

which follows from the properties of $*$. These are, however, the defining relations for the Clifford algebra $C(3, 1) \simeq \mathrm{End}_{\mathbb{R}}(\mathbb{C}^2)$ (see [8]). Thus,

$$\gamma_\mu = *\sigma_\mu \tag{26}$$

which is consistent with the fact that $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$ is multiplication by i. In terms of the real basis of $W(3, 1)$, the gamma matrices are given by

$$\begin{aligned} \gamma_0 &= \begin{pmatrix} -J_2 & 0 \\ 0 & J_2 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} \sigma_3{}^{R} & 0 \\ 0 & -\sigma_3{}^{R} \end{pmatrix} \\ \gamma_2 &= \begin{pmatrix} 0 & -1_2 \\ -1_2 & 0 \end{pmatrix} \qquad \gamma_3 = \begin{pmatrix} -\sigma_1{}^{R} & 0 \\ 0 & \sigma_1{}^{R} \end{pmatrix} \end{aligned} \tag{27}$$

which have definite parity; here, $\sigma_1{}^{R}$, and $\sigma_3{}^{R}$, denote the real 2×2 matrices whose entries are given exactly as in (25) above, but thought to be associated with real maps $\mathbb{R}^2 \to \mathbb{R}^2$. In particular, criterion 15 is again satisfied and the representation of $D(3, 1)$ induced by these γs preserves orthogonal complements. In terms of real 4×4 matrices, the action of the generators β_μ is given by

$$\begin{aligned} \beta_0 \begin{pmatrix} A & C \\ B & D \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} [A, J_2] & -\{C, J_2\} \\ \{B, J_2\} & -[D, J_2] \end{pmatrix} \\ \beta_1 \begin{pmatrix} A & C \\ B & D \end{pmatrix} &= \frac{1}{2}\begin{pmatrix} \{A, \sigma_3\} & -[C, \sigma_3] \\ [B, \sigma_3] & -\{D, \sigma_3\} \end{pmatrix} \\ \beta_2 \begin{pmatrix} A & C \\ B & D \end{pmatrix} &= -\frac{1}{2}\begin{pmatrix} B+C & A+D \\ A+D & B+C \end{pmatrix} \\ \beta_3 \begin{pmatrix} A & C \\ B & D \end{pmatrix} &= -\frac{1}{2}\begin{pmatrix} \{A, \sigma_1\} & -[C, \sigma_1] \\ [B, \sigma_1] & -\{D, \sigma_1\} \end{pmatrix} \end{aligned} \tag{28}$$

where A, B, C and D are 2×2 blocks, and $[X,Y]$ (respectively, $\{X,Y\}$) denotes the commutator (respectively, anticommutator) of the matrices X and Y. On the other hand, in terms of quaternionic 2×2 matrices, $Q = A + B\boldsymbol{i} + C\boldsymbol{j} + D\boldsymbol{k}$, the action is

$$\begin{aligned} &\beta_0(Q) = -\tfrac{1}{2}(\boldsymbol{i}Q\boldsymbol{i}J_2 - J_2\boldsymbol{j}Q\boldsymbol{j}) \qquad \beta_1(Q) = -\tfrac{1}{2}(\boldsymbol{i}Q\boldsymbol{i}\sigma_3 + \sigma_3\boldsymbol{j}Q\boldsymbol{j}) \\ &\beta_2(Q) = -\tfrac{1}{2}(\boldsymbol{i}Q\boldsymbol{i}\boldsymbol{j} + \boldsymbol{i}\boldsymbol{j}Q\boldsymbol{j}) \qquad \beta_3(Q) = \tfrac{1}{2}(\boldsymbol{i}Q\boldsymbol{i}\sigma_1 + \sigma_1\boldsymbol{j}Q\boldsymbol{j}). \end{aligned} \tag{29}$$

In these expressions, the σ_μs are the Pauli matrices though inside $\boldsymbol{H}^{2\times 2}$. Note that the common kernel of the β_μs is defined by the conditions

$$\begin{aligned} &D = -A \qquad C = -B \qquad [A, J_2] = 0 = \{A, \sigma_1\} = \{A, \sigma_3\} \\ &\{B, J_2\} = 0 = [B, \sigma_1] = [B, \sigma_3]. \end{aligned}$$

These further imply $B = 0$, and $A = aJ_2$, with $a \in \boldsymbol{R}$. Thus, the common kernel is one-dimensional and therefore carries the trivial one-dimensional representation of $D(3,1)$ (see remark 7). Moreover, its orthogonal complement within the skew-symmetric matrices with respect to the orthogonal structure given in (8) is the set of matrices of the form

$$X = \begin{pmatrix} bJ_2 & B \\ -B^t & bJ_2 \end{pmatrix} \qquad b \in \boldsymbol{R} \text{ and } B \in \boldsymbol{R}^{2\times 2} \text{ arbitrary} \tag{30}$$

and carries the irreducible five-dimensional representation. The irreducibility follows by showing that it is a cyclic submodule, which in turn follows from the explicit action of the β_μs. Similarly, the subspace of symmetric matrices is irreducible for the same reason: the explicit action of the β_μs leads one to the conclusion that it is a cyclic submodule; it is the carrier of the ten-dimensional representation of $D(3,1)$.

4.3. *Case $q = 2$*

We shall continue using the notation introduced in [8]. Thus, $\{e_-, e_0, e_1, e_2, e_3, e_+\}$ is an orthonormal basis of $\boldsymbol{R}^{4,2}$, with $-B(e_-, e_-) = -B(e_0, e_0) = 1 = B(e_j, e_j) = B(e_+, e_+)$, for $j = 1$, 2 and 3. The Clifford module, $W(4,2)$, is now isomorphic to $\boldsymbol{R}^8$, and when viewed as the complex space $\boldsymbol{C}^4$ via γ_7, it is endowed with a Hermitian structure of signature $(2,2)$, since $Spin(4,2)_e \simeq SU(2,2)$. There is again a natural choice for the antilinear map a_2 appearing in the decomposition (3); namely, the (up to a constant) unique map that intertwines the two Weyl components in $\boldsymbol{C}^4$:

$$a_2\colon \boldsymbol{C}^4 \to \boldsymbol{C}^4 \qquad \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \mapsto \begin{pmatrix} *u_2 \\ -*u_1 \end{pmatrix} \qquad u_1, u_2 \in \boldsymbol{C}^2. \tag{31}$$

This is the charge conjugation map at the vector space level (see [9]). The gamma matrices this time are given by [8]

$$\gamma_\pm = \begin{pmatrix} 0 & * \\ \mp * & 0 \end{pmatrix} \qquad \gamma_0 = \begin{pmatrix} * & 0 \\ 0 & -* \end{pmatrix} \qquad \gamma_j = \begin{pmatrix} -*\sigma_j & 0 \\ 0 & -*\sigma_j \end{pmatrix} \qquad j = 1, 2, 3. \tag{32}$$

They satisfy the Clifford algebra relations,

$$\begin{aligned} &\gamma_+\gamma_- + \gamma_-\gamma_+ = 0 \qquad \gamma_\pm\gamma_\mu + \gamma_\mu\gamma_\pm = 0 \qquad {\gamma_-}^2 = {\gamma_0}^2 = -1 \\ &{\gamma_+}^2 = {\gamma_j}^2 = 1 \qquad j = 1, 2, 3. \end{aligned}$$

Moreover, thinking of the maps $*$, and σ_μ of C^2 into itself in terms of the underlying four-dimensional real space, we have

$$
\begin{aligned}
* &= \begin{pmatrix} -J_2 & 0 \\ 0 & J_2 \end{pmatrix} \qquad \sigma_0 = \begin{pmatrix} 1_2 & 0 \\ 0 & 1_2 \end{pmatrix} \qquad \sigma_1 = \begin{pmatrix} {\sigma_1}^R & 0 \\ 0 & {\sigma_1}^R \end{pmatrix} \\
\sigma_2 &= \begin{pmatrix} 0 & -J_2 \\ J_2 & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} {\sigma_3}^R & 0 \\ 0 & {\sigma_3}^R \end{pmatrix}
\end{aligned} \tag{33}
$$

where ${\sigma_1}^R$ and ${\sigma_3}^R$ are as in (27). All these matrices have definite symmetry, as

$$
^t = - \qquad {\sigma_\mu}^t = \sigma_\mu \qquad \mu = 0, 1, 2, 3.
$$

Therefore, the same is true for the gamma matrices (32). Hence, the representation of $D(4, 2)$ obtained as in proposition 11 preserves orthogonal complements (see criterion 15). Finally, the action of the β_μs on the space of 8×8 real matrices is ($j = 1$, 2, and 3)

$$
\begin{aligned}
\beta_0 \begin{pmatrix} A & C \\ B & D \end{pmatrix} &= \frac{1}{2} \begin{pmatrix} -[A, *] & \{C, *\} \\ -\{B, *\} & [D, *] \end{pmatrix} \\
\beta_j \begin{pmatrix} A & C \\ B & D \end{pmatrix} &= -\frac{1}{2} \begin{pmatrix} \{A, *\sigma_j\} & \{C, *\sigma_j\} \\ \{B, *\sigma_j\} & \{D, *\sigma_j\} \end{pmatrix}
\end{aligned} \tag{34}
$$

(where use has been made of the fact that when $*$ and σ_j are thought of as real 4×4 matrices $*\sigma_j + \sigma_j* = 0$), and

$$
\beta_\pm \begin{pmatrix} A & C \\ B & D \end{pmatrix} = \frac{1}{2} \begin{pmatrix} *B - C* & \pm(A * \pm * D) \\ \mp(*A \pm D*) & \pm(B * - * C) \end{pmatrix}. \tag{35}
$$

In terms of quaternionic 4×4 matrices, we have

$$
\begin{aligned}
&\beta_0(Q) = \tfrac{1}{2}(\boldsymbol{i}Q\boldsymbol{i} * - * \boldsymbol{j}Q\boldsymbol{j}) \qquad \beta_j(Q) = -\tfrac{1}{2}\{Q, *\sigma_j\} \qquad j = 1, 2, 3 \\
&\beta_+(Q) = -\tfrac{1}{2}(*\boldsymbol{i}Q - Q\boldsymbol{j}*) \qquad \beta_-(Q) = \tfrac{1}{2}(*\boldsymbol{ik}Q\boldsymbol{k} - \boldsymbol{i}Q\boldsymbol{ij}*).
\end{aligned} \tag{36}
$$

From these expressions one determines the common kernel of the β_μs: from $\beta_\pm(Q) = 0$ one obtains $*A = \mp D*$, and $*B = C*$, which immediately yields $A = 0 = D$. With this information, $\beta_0(Q) = 0$ implies $*C*^{-1} = -C$ with an identical equation for B. Using this, one finds from $\beta_j(Q) = 0$ that $[C, \sigma_j] = 0$ ($j = 1$, 2 and 3), and similarly for B. It then follows that C is a complex multiple of the 2×2 identity matrix. Since $*C*^{-1} = -C$, it follows that in fact, $C = \mathrm{i}c1_2$, with $c \in \boldsymbol{R}$. Reverting to real matrices, it follows that $C = cJ_4$ (cf (22)) and hence, $B = C^t$. Thus, the matrices of the form,

$$
\xi = \begin{pmatrix} 0 & C \\ C^t & 0 \end{pmatrix} \qquad C = c \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix} \qquad c \in \boldsymbol{R} \tag{37}
$$

carry the trivial, one-dimensional representation of $D(4, 2)$; this time occurring inside the space of symmetric matrices, $S^2(\boldsymbol{R}^8)$. Its orthogonal complement with respect to the form (8) is the 35-dimensional irreducible representation; it is a cyclic submodule. On the other hand, the subspace, $A^2(\boldsymbol{R}^8)$, of skew-symmetric matrices breaks up into two irreducible representations of dimensions 7, and 21, respectively. The seven-dimensional module consists of all matrices of the form,

$$\xi = \begin{pmatrix} b_1 & B+\beta & 0 & \delta \\ -(B^t+\beta) & b_2 & -\delta & 0 \\ 0 & \delta & -b_1 & -(B-\beta) \\ -\delta & 0 & (B^t-\beta) & -b_2 \end{pmatrix} \tag{38}$$

with b_1, and b_2 in $A^2(\mathbf{R}^2)$, β, and δ real multiples of the 2×2 identity matrix and B a zero-trace 2×2 matrix. The 21-dimensional submodule is defined by the orthogonal complement of these matrices, i.e. all matrices of the form

$$\zeta = \begin{pmatrix} b_3 & E+\nu & R & C-\mu \\ -(E^t+\nu) & b_4 & D-\mu & S \\ -R^t & -(D^t-\mu) & b_3 & E-\nu \\ -(C^t-\mu) & -S^t & -(E^t-\nu) & b_4 \end{pmatrix} \tag{39}$$

with b_3, and b_4 in $A^2(\mathbf{R}^2)$, μ, and ν real multiples of the 2×2 identity matrix, C, D and E traceless, and R and S, arbitrary 2×2 matrices.

5. Orthogonal structure of the DK modules

What remains now is to determine the signature of the orthogonal form (8) to completely elucidate the geometric structure of the irreducible DK modules just described (see remark 5). The case $q = 0$ is extremely easy, since $(\cdot,\cdot)_0$ has signature $(4, 0)$ and corresponds to the standard, positive-definite, orthogonal structure on matrices: $(\xi, \zeta) \mapsto \mathrm{Tr}(\xi\zeta^t)$. It is therefore restricted to positive definite forms on the spaces of symmetric and skew-symmetric matrices.

Now, for the case $q = 1$, we compute explicitly the quadratic form on $S^2(\mathbf{R}^4)$, and $A^2(\mathbf{R}^4)$. First of all, the orthogonal structure on $W(3,1) \otimes W(3,1)$ arises as the tensor product of two symplectic forms. In fact, $W(3,1)$ is equipped with the geometry of $Spin(3,1)$ whose identity component is $SL(2,\mathbf{C})$. After fixing a basis of $W(3,1)$, the symplectic form is the one associated with the matrix J_2. This yields an isomorphism $SL(2,\mathbf{C}) \simeq Sp(2,\mathbf{C})$. Finally, by splitting this symplectic form into its real and imaginary parts, one defines symplectic forms, $\mathrm{Re}\, J_2$, and $\mathrm{Im}\, J_2$ on the underlying four-dimensional, real space. By looking at either, $\mathrm{Re}\, J_2 \otimes \mathrm{Re}\, J_2$, or $\mathrm{Im}\, J_2 \otimes \mathrm{Im}\, J_2$, one finds that the signature of the orthogonal structure on the tensor product space is $(8, 8)$. The details of these particular results will be substituted by proving instead the following two general propositions:

Proposition 16. There are inclusions,

$$\mathrm{Re}\colon Sp(2n,\mathbf{C}) \hookrightarrow Sp(4n,\mathbf{R}) \qquad \text{and} \qquad \mathrm{Im}\colon Sp(2n,\mathbf{C}) \hookrightarrow Sp(4n,\mathbf{R}).$$

Furthermore, the two are related via right multiplication by a non-singular matrix,

$$\mathrm{Re}(X) = \mathrm{Im}(X) J_{4n}$$

where J_{4n} is as in (22*b*)

Proof. Let ω be the standard symplectic form on $\boldsymbol{C}^{2n}$. Then,

$$\begin{aligned}\omega\left(\begin{pmatrix}z_1\\z_2\end{pmatrix},\begin{pmatrix}\zeta_1\\\zeta_2\end{pmatrix}\right) &= (z_1{}^t \quad z_2{}^t)\begin{pmatrix}0 & -1_n\\1_n & 0\end{pmatrix}\begin{pmatrix}\zeta_1\\\zeta_2\end{pmatrix} = z_2\cdot\zeta_1 - z_1\cdot\zeta_1\\ &= (x_2+\mathrm{i}y_2)\cdot(\xi_1+\mathrm{i}\eta_1)-(x_1+\mathrm{i}y_1)\cdot(\xi_2+\mathrm{i}\eta_2)\\ &= (x_2\cdot\xi_1 - y_2\cdot\eta_1 - x_1\cdot\xi_2 + y_1\cdot\eta_2)+\mathrm{i}(x_2\cdot\eta_1+y_2\cdot\xi_1-x_1\cdot\eta_2-y_1\cdot\xi_2)\end{aligned}$$

where, x_i, y_i, ξ_i, $\eta_i \in \boldsymbol{R}^n$, for $i = 1, 2$. Hence,

$$\mathrm{Re}\,\omega \longleftrightarrow \begin{pmatrix}J_{2n} & 0\\0 & -J_{2n}\end{pmatrix} \qquad \text{and} \qquad \mathrm{Im}\,\omega \longleftrightarrow \begin{pmatrix}0 & J_{2n}\\J_{2n} & 0\end{pmatrix}.$$

Both are skew-symmetric and therefore, define symplectic forms on the underlying real space $\boldsymbol{R}^{4n}$. Moreover, the two are related as stated. Finally, the inclusions, $Sp(2n,\boldsymbol{C}) \hookrightarrow Sp(4n,\boldsymbol{R})$ and $Sp(2n,\boldsymbol{C}) \hookrightarrow Sp(4n,\boldsymbol{R})$, follow by comparing real and imaginary parts of the equality $\omega(gu, gv) = \omega(u, v)$, which holds for all u and v in $\boldsymbol{C}^{2n}$ and $g \in Sp(2n,\boldsymbol{C})$. □

Proposition 17. Let $(\boldsymbol{R}^{4n}, \mathrm{Re}\,\omega)$ be the symplectic real space obtained from the symplectic form ω on $\boldsymbol{C}^{2n}$. Identify $\boldsymbol{R}^{4n}\otimes\boldsymbol{R}^{4n}$ with the space of $4n\times 4n$ matrices. Let $(\cdot,\cdot)$ be the orthogonal form $\mathrm{Re}\,\omega\otimes\mathrm{Re}\,\omega$. Then, the signature of $(\cdot,\cdot)$ restricted to the subspace of symmetric (respectively, skewsymmetric) matrices is $(4n^2, 2n(2n+1))$ (respectively, $(4n^2, 2n(2n-1))$). The same is true if $\mathrm{Re}\,\omega$ is replaced by $\mathrm{Im}\,\omega$.

Proof. Let ω_0 be the matrix corresponding to $\mathrm{Re}\,\omega$. It follows from $A2$ in the appendix that, for all $\xi\in\boldsymbol{R}^{4n\times 4n}$, $(\xi,\xi) = \mathrm{Tr}(\xi(\omega_0\xi\omega_0{}^t)^t)$. In particular, for all symmetric matrices, $\xi\in S^2(\boldsymbol{R}^4 n)$,

$$(\xi,\xi) = -\,\mathrm{Tr}(\xi\omega_0\xi\omega_0).$$

A plus sign appears on the right-hand side for skew-symmetric matrices. Thus, writing

$$\xi = \begin{pmatrix}A_0 & B_1 & B_2 & B_3\\ B_1{}^t & A_1 & C_3 & C_2\\ B_2{}^t & C_3{}^t & A_2 & C_1\\ B_3{}^t & C_2{}^t & C_1{}^t & A_3\end{pmatrix} \in S^2(\boldsymbol{R}^{4n\times 4n})$$

with $A_i\in S^2(\boldsymbol{R}^n)$, and B_i, and C_i arbitrary, we have

$$(\xi,\xi) = 4\,\mathrm{Tr}(B_3C_3{}^t - B_2C_2{}^t) + 2\,\mathrm{Tr}(A_0A_1 + A_2A_3) - 2\,\mathrm{Tr}(B_1^2 + C_1^2). \quad (40)$$

The computation of the signature is reduced to knowing the signature of the quadratic forms: $\boldsymbol{R}^{n\times n}\ni X\mapsto \{X,X\}_n = \mathrm{Tr}(X^2)$ (see lemma 18); and $\langle X,X\rangle_n = \mathrm{Tr}(XX^t)$; the latter being the standard, positive-definite form on matrices. Once this is known, the contributions to the signature of $(\cdot,\cdot)$ can be easily counted from (40); e.g., the contribution of $\mathrm{Tr}(B_3C_3{}^t)$ is found by rewriting B_3 (respectively, C_3) in the form $X+Y$ (respectively, $X-Y$), with X, and Y arbitrary, with similar substitutions for the other crossed terms. □

Lemma 18. The quadratic form $\mathbb{R}^{n\times n} \ni X \mapsto \{X, X\}_n = \mathrm{Tr}(X^2)$ is positive definite on $S^2(\mathbb{R}^n)$, and negative definite on $A^2(\mathbb{R}^n)$. Furthermore,

$$\mathrm{sgn}\{\cdot,\cdot\}_n = \left(\tfrac{1}{2}n(n+1), \tfrac{1}{2}n(n-1)\right).$$

Proof. Recursively, one finds that

$$\mathrm{sgn}\{\cdot,\cdot\}_n = (p_n, q_n) = (n + p_{n-1}, (n-1) + q_{n-1})$$

and the statement follows from this by induction. □

Corollary 19. (i) The restriction of $(\cdot,\cdot)_1$ to the irreducible $D(3,1)$-module, $S^2(\mathbb{R}^4)$ has signature $(4,6)$;

(ii) its restriction to the invariant submodule, $A^2(\mathbb{R}^4)$, has signature $(4,2)$; and,

(iii) it is negative definite on the one-dimensional trivial submodule 3 (20a).

Proof. (i) and (ii) follow from proposition 17. What remains is (iii), which is a straightforward computation that we safely leave to the reader. □

Finally, for the case $q = 2$, we summarize the results in the following,

Proposition 5. (i) $(\cdot,\cdot)_2$ is negative definite on the one-dimensional, $D(4,2)$ submodule (37);

(ii) its restriction to the irreducible, seven-dimensional submodule (38), has signature $(4,3)$;

(iii) its restriction to the irreducible 21-dimensional submodule (39) has signature $(12,9)$; and,

(iv) its restriction to the irreducible 35-dimensional representation has signature $(16,19)$.

Proof. (i) Let ξ be as in (37). Then,

$$(\xi,\xi)_2 = -\mathrm{Tr}(\xi b_2 \xi b_2) = -8c^2.$$

(ii) Now, let ξ be as in (38). Then,

$$(\xi,\xi)_2 = \mathrm{Tr}(\xi b_2 \xi b_2) = 4\,\mathrm{Tr}\,\delta^2 - 4\,\mathrm{Tr}\,\beta^2 + 4\,\mathrm{Tr}(B^t B) + 2\,\mathrm{Tr}\,{b_1}^2 + 2\,\mathrm{Tr}\,{b_2}^2.$$

Now, the quadratic form, $a \mapsto \mathrm{Tr}\,a^2$, for $a \in A^2(\mathbb{R}^2)$, is negative definite by lemma 18. Similarly, the form $\mathbb{R}^{2\times 2} \ni Z \mapsto \mathrm{Tr}(Z^t Z) \in \mathbb{R}$, is positive-definite and, so is its restriction to the subspace of zero-trace matrices. It follows that the signature of $(\cdot,\cdot)_2$ on the 7-dimensional irreducible subspace is $(4,3)$, as claimed.

(iii) This time let ξ be as in (39). Then,

$$\begin{aligned}\mathrm{Tr}(\xi b_2 \xi b_2) = {} & 2\,\mathrm{Tr}\,R^2 + 2\,\mathrm{Tr}\,S^2 - 4\,\mathrm{Tr}(E^t E) - 4\,\mathrm{Tr}(CD) \\ & - 4\,\mathrm{Tr}(\mu^2) + 4\,\mathrm{Tr}(\nu^2) - 2\,\mathrm{Tr}\,{b_3}^2 - 2\,\mathrm{Tr}\,{b_4}^2\end{aligned}$$

and the signature can be determined using lemma 18, and counting the various contributions; e.g., the contribution of the term $\mathrm{Tr}(CD)$ is found by rewriting C and D as $X + Y$, and $X - Y$ respectively. The conditions $\mathrm{Tr}\,C = 0 = \mathrm{Tr}\,D$ translate into $\mathrm{Tr}\,X = 0 = \mathrm{Tr}\,Y$. Thus,

$$\mathrm{Tr}(CD) = \mathrm{Tr}((X+Y)(X-Y)) = \mathrm{Tr}\,X^2 - \mathrm{Tr}\,Y^2$$

and the results of lemma 18 can be applied directly.

(iv) The 35-dimensional module is the orthogonal complement in $S^2(\mathbb{R}^8)$ of the subspace (37). It therefore consists of all matrices of the form

$$\xi = \begin{pmatrix} S_1 & A & C & X+\alpha \\ A^t & S_2 & Y+\alpha & D \\ C^t & Y^t+\alpha & S_3 & B \\ X^t+\alpha & D^t & B^t & S_4 \end{pmatrix}$$

where all the blocks are 2×2 matrices. More specifically, A, B, C and D are arbitrary X and Y are traceless; S_1, S_2, S_3 and S_4 are symmetric and α is a multiple of the identity. Then,

$$\begin{aligned}(\xi,\xi)_2 = -\operatorname{Tr}(\xi b_2 \xi b_2) &= 4\operatorname{Tr}(XY) + 4\operatorname{Tr}(\alpha^2) + 2\operatorname{Tr}(S_1S_3 + S_2S_4) \\ &\quad - 4\operatorname{Tr}(AB^t) - 2\operatorname{Tr}(C^2 + D^2)\end{aligned}$$

and a simple counting of the various contributions yield the result. □

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Appendix

Let (V_1, B_1) and (V_2, B_2) be finite-dimensional real vector spaces with non-degenerate bilinear forms of definite parity (i.e. either symmetric or skew-symmetric). Let (V, B) be the pair obtained from (V_1, B_1) and (V_2, B_2) by setting $V = V_1 \otimes V_2$ and $B = B_1 \otimes B_2$. Thus,

$$B(u_1 \otimes u_2, v_1 \otimes v_2) = B_1(u_1, v_1)B_2(u_2, v_2)$$

on decomposables. Then, $B_1 \otimes B_2$ has definite parity: it is symmetric whenever B_1 and B_2 have the same parity and skew-symmetric otherwise. Furthermore,

Proposition A1. The bilinear form $B_1 \otimes B_2$ is non-degenerate. It is symplectic or orthogonal according to the scheme shown in table A1:

Table A1. Tensor product of symplectic and orthogonal spaces.

(V_1, B_1)	(V_2, B_2)	$(V_1 \otimes V_2, B_1 \otimes B_2)$
Orthogonal; $\mathrm{sgn} = (p_1, q_1)$	Orthogonal; $\mathrm{sgn} = (p_2, q_2)$	Orthogonal; $\mathrm{sgn} = (p_1p_2 + q_1q_2, p_1q_2 + q_1p_2)$
Orthogonal; $\mathrm{sgn} = (p_1, q_1)$	Symplectic; $\dim = 2n$	Symplectic; $\dim = 2n(p_1 + q_1)$
Symplectic; $\dim = 2m$	Symplectic; $\dim = 2n$	Orthogonal; $\mathrm{sgn} = (2mn, 2mn)$

Proof. The proof consists of computing the matrix elements of $B_1 \otimes B_2$ relative to some special choices of bases in V_1 and V_2.

(i) Let (V_1, B_1) and (V_2, B_2) be orthogonal spaces with $\operatorname{sgn} B_1 = (p_1, q_1)$ and $\operatorname{sgn} B_2 = (p_2, q_2)$. Choose orthonormal bases $\{e_i, e_{p_1+\mu}\}$ and $\{f_r, f_{p_2+\sigma}\}$ of V_1, and V_2, respectively, with $B_1(e_i, e_j) = \delta_{ij}$, for $1 \leqslant i, j \leqslant p_1$, and $B_1(e_{p_1+\mu}, e_{p_1+\nu}) = -\delta_{\mu\nu}$, for $1 \leqslant \mu, \nu \leqslant q_1$. (Similarly, $B_2(f_r, f_s) = \delta_{rs}$, for $1 \leqslant r, s \leqslant p_2$, and $B_2(f_{p_2+\sigma}, f_{p_2+\tau}) = -\delta_{\sigma\tau}$, for $1 \leqslant \sigma, \tau \leqslant q_2$.) Then, $\{e_i \otimes f_r, e_{p_1+\mu} \otimes f_{p_2+\sigma}, e_i \otimes f_{p_2+\sigma}, e_{p_1+\mu} \otimes f_r\}$ is an orthonormal basis for $V_1 \otimes V_2$. In terms of this basis, the matrix of $B_1 \otimes B_2$ takes the form

$$\begin{array}{r|cccc|}
 & e_j \otimes f_s & e_{p_1+\nu} \otimes f_{p_2+\tau} & e_j \otimes f_{p_2+\tau} & e_{p_1+\nu} \otimes f_s \\
e_i \otimes f_r & 1 & 0 & 0 & 0 \\
e_{p_1+\mu} \otimes f_{p_2+\sigma} & 0 & 1 & 0 & 0 \\
e_i \otimes f_{p_2+\sigma} & 0 & 0 & -1 & 0 \\
e_{p_1+\mu} \otimes f_r & 0 & 0 & 0 & -1
\end{array}$$

where the diagonal blocks have sizes $p_1p_2 \times p_1p_2$, $q_1q_2 \times q_1q_2$, $p_1q_2 \times p_1q_2$, and $p_2q_1 \times p_2p_1$, respectively.

(ii) Now, let (V_1, B_1) be orthogonal with $\operatorname{sgn} B_1 = (p_1, q_1)$, and let (V_2, B_2) be symplectic of dimension $2n$. Let $\{e_i, e_{p_1+\mu}\}$ be as in (i), and let $\{f_r, f_{n+r} \mid 1 \leqslant r \leqslant n\}$ be a symplectic basis for V_2; i.e.

$$B_2(f_{n+r}, f_s) = \delta_{rs} = -B_2(f_r, f_{n+s}) \qquad B_2(f_r, f_s) = 0 = B_2(f_{n+r}, f_{n+s}).$$

Then, $\{e_i \otimes f_r, e_i \otimes f_{n+r}, e_{p_1+\mu} \otimes f_r, e_{p_1+\mu} \otimes f_{n+r}\}$ is a basis of $V_1 \otimes V_2$, with respect to which the matrix of $B_1 \otimes B_2$ is skew-symmetric and invertible. It is given by

$$\begin{array}{r|cccc|}
 & e_j \otimes f_s & e_j \otimes f_{n+s} & e_{p_1+\nu} \otimes f_s & e_{p_1+\nu} \otimes f_{n+s} \\
e_i \otimes f_r & 0 & -1 & 0 & 0 \\
e_i \otimes f_{n+r} & 1 & 0 & 0 & 0 \\
e_{p_1+\mu} \otimes f_r & 0 & 0 & 0 & 1 \\
e_{p_1+\mu} \otimes f_{n+r} & 0 & 0 & -1 & 0
\end{array}$$

Here, the first two diagonal blocks are of size $np_1 \times np_1$, while the last two have size $nq_1 \times nq_1$.

(iii) Now let (V_1, B_1) and (V_2, B_2) be symplectic spaces of dimensions $2m$ and $2n$, respectively, and let $\{e_i, e_{m+i} \mid 1 \leqslant i \leqslant m\}$ and $\{f_r, f_{n+r} \mid 1 \leqslant r \leqslant n\}$ be symplectic bases for them. Then, $\{e_i \otimes f_r, e_i \otimes f_{n+r}, e_{m+i} \otimes f_r, e_{m+i} \otimes f_{n+r}\}$ is an orthonormal basis for $V_1 \otimes V_2$ of signature $(2nm, 2nm)$. In fact, in terms of this basis, the matrix of $B_1 \otimes B_2$ takes the form

$$\begin{array}{r|cccc|}
 & e_j \otimes f_s & e_j \otimes f_{n+s} & e_{m+j} \otimes f_s & e_{m+j} \otimes f_{n+s} \\
e_i \otimes f_r & 0 & 0 & 0 & 1 \\
e_i \otimes f_{n+r} & 0 & 0 & -1 & 0 \\
e_{m+i} \otimes f_r & 0 & -1 & 0 & 0 \\
e_{m+i} \otimes f_{n+r} & 1 & 0 & 0 & 0
\end{array}$$

where the entries are $nm \times nm$ blocks. To elucidate the signature of this matrix—which we shall denote by B—we change the basis in $V_1 \otimes V_2$ by means of $g \in GL(4mn, R)$ of the form,

$$g = \begin{pmatrix} A & 0 & 0 & B \\ 0 & a & b & 0 \\ 0 & c & d & 0 \\ C & 0 & 0 & D \end{pmatrix}$$

where the entries are again $nm \times nm$ blocks. Thus,

$$gBg^t = \begin{pmatrix} BA^t + AB^t & 0 & 0 & BC^t + AD^t \\ 0 & -(ba^t + ab^t) & -(bc^t + ad^t) & 0 \\ 0 & -(da^t + cb^t) & -(dc^t + cd^t) & 0 \\ DA^t + CB^t & 0 & 0 & DC^t + CD^t \end{pmatrix}.$$

We now choose the blocks of g as follows

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \qquad \text{and} \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}.$$

With these choices, $gBg^t = \mathrm{diag}\{1_{nm}, 1_{nm}, -1_{nm}, -1_{nm}\}$, which completes the proof. □

Remark A2. When $V_1 \otimes V_2$ is identified with the space of $\dim V_1 \times \dim V_2$ real matrices, via

$$\xi = \sum_{\substack{i=1 \\ r=1}}^{\substack{i=\dim V_1 \\ r=\dim V_2}} \xi_{ir} e_i \otimes f_r \longleftrightarrow \xi = (\xi_{ir})$$

for some choices of bases $\{e_i\}$ and $\{f_r\}$, the orthogonal or symplectic structure $B = B_1 \otimes B_2$ is given by

$$B(\xi, \zeta) = \mathrm{Tr}(\xi (b_1 \zeta b_2{}^t)^t)$$

where b_1 and b_2 are the matrices of B_1 and B_2, respectively.

Now, the proof of the following result is essentially the same as the proof of the orthogonal–orthogonal case of proposition A1.

Proposition A3. (Tensor product of Hermitian spaces.) Let (W_1, H_1), and (W_2, H_2) be two Hermitian spaces with signatures (p_1, q_1) and (p_2, q_2), respectively. Let (W, H) be the pair $W = W_1 \otimes W_2$ and $H = H_1 \otimes H_2$. Then, H is a Hermitian form on W of signature $(p_1 p_2 + q_1 q_2, p_1 q_2 + p_2 q_1)$. Moreover, when $W_1 \otimes W_2$ is identified with the space of $\dim W_1 \times \dim W_2$-matrices after some definite choice of bases, the Hermitian structure H is given by

$$H(\xi, \zeta) = \mathrm{Tr}(\xi (h_1 \bar{\zeta} h_2{}^t)^t)$$

where h_1, and h_2 are the matrices of H_1, and H_2, respectively, with respect to the chosen bases of W_1, and W_2.

Proposition A4. Let (W_1, H_1), (W_2, H_2), and (W, H) be as in the proposition above. The real and imaginary parts of H define orthogonal and symplectic structures, respectively, on the underlying real space, $(W_1 \otimes W_2)_R$, of $W_1 \otimes W_2$, according to

$$\operatorname{Re} H \longleftrightarrow \text{ Orthogonal of signature } (2(p_1p_2 + q_1q_2), 2(p_1q_2 + p_2q_1))$$

$$\operatorname{Im} H \longleftrightarrow \text{ Symplectic on (real) dimension } 2(\dim W_1)(\dim W_2).$$

Proof. Let $\{e_i, e_{p_1+\mu}\}$ and $\{f_r, f_{p_2+\sigma}\}$ be orthonormal bases for W_1 and W_2, respectively. Then, $\{e_i \otimes f_r, e_{p_1+\mu} \otimes f_{p_2+\sigma}, e_i \otimes f_{p_2+\sigma}, e_{p_1+\mu} \otimes f_r\}$ is an orthonormal basis of $W_1 \otimes W_2$, with respect to which the matrix h of the Hermitian form H has the block form

$$h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Under the identification,

$$\boldsymbol{C}^{(p_1+q_1)(p_2+q_2)} \ni (x_1 + \mathrm{i}y_1, \ldots, x_4 + \mathrm{i}y_4) \leftrightarrow (x_1, \ldots, x_4, y_1, \ldots, y_4) \in \boldsymbol{R}^{2(p_1+q_1)(p_2+q_2)}$$

(with x_1, y_1 in $\boldsymbol{R}^{p_1q_1}$, x_2, y_2 in $\boldsymbol{R}^{p_2q_2}$, x_3, y_3 in $\boldsymbol{R}^{p_1q_2}$ and x_4, y_4 in $\boldsymbol{R}^{p_2q_1}$), the matrices of the real and imaginary parts of H are, respectively, given by

$$\operatorname{Re} H = \begin{pmatrix} 1_{p_1q_1+p_2q_2} & 0 & 0 & 0 \\ 0 & -1_{p_1q_2+p_2q_1} & 0 & 0 \\ 0 & 0 & 1_{p_1q_1+p_2q_2} & 0 \\ 0 & 0 & 0 & -1_{p_1q_2+p_2q_1} \end{pmatrix}$$

and

$$\operatorname{Im} H = \begin{pmatrix} 0 & 0 & 1_{p_1q_1+p_2q_2} & 0 \\ 0 & 0 & 0 & -1_{p_1q_2+p_2q_1} \\ -1_{p_1q_1+p_2q_2} & 0 & 0 & 0 \\ 0 & 1_{p_1q_2+p_2q_1} & 0 & 0 \end{pmatrix}$$

from which the assertion follows.

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